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UNIVERSAL VELOCITY AND REYNOLDS-STRESS PROFILES IN A TRANSPIRED TURBULENT BOUNDARY LAYER ON A PLATE†

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The flow of an incompressible fluid in a turbulent boundary layer on a plate with distributed transpiration is considered. Closure conditions are obtained in the form of universal dependences of the shear stress and other components of the Reynolds tensor on the gradient of the mean streamwise velocity, which can only be found on the basis of data that are known for a flow over an impermeable plate. As a result of the asymptotic solution of the problem for high Reynolds numbers, similarity laws for the velocity profiles and turbulent stresses are established, which enable one to calculate these quantities using data which are known for an impermeable plate.

The well-known analysis of the flow in a turbulent boundary layer on an impermeable plate, which was carried out for the first time in [1–3] using dimensional analysis (see also [4]), is obviously very restricted with respect to the number of governing parameters of the problem and does not yield any interesting conclusions when there is a non-zero transverse velocity on the wall. In this case, however, a universal functional relation can be established between the turbulent shear stress and the gradient of the mean velocity [5, 6], and the use of this relation, together with the equations of motion, enables one to solve the problem.

The similarity rules for the velocity and temperature in the near-wall region of a boundary layer, which extend the logarithmic law to the case of a transpired wall were obtained in this way [5] and the parameters in the asymptotic case of boundary layer with suction were calculated in [6].

The further development of this method [5, 6] in the present paper has enabled us to solve analytically the problem for a boundary layer on a plate with moderate injection or suction, and to obtain universal representations of the velocity profile and components of the Reynolds tensor which hold everywhere outside the viscous sublayer.

Some of the results presented here were obtained earlier in [7] using a less general formulation.

1. FORMULATION OF THE PROBLEM

Consider the incompressible turbulent boundary layer flow over a smooth flat plate with a constant free-stream velocity U_e at the outer edge of the boundary layer and with a constant transpiration velocity υ_w directed along the normal to the plate surface. We will assume that turbulent flow develops immediately from the leading edge of the plate. We will also assume that the level of turbulent fluctuations in the free stream is negligibly small and has no effect on the flow in the boundary layer.

The gradient of the mean longitudinal velocity and the turbulent shear stress are functions of the Cartesian x and y coordinates and of the governing parameters of the problem:

$$\partial u/\partial y = F_1(x, y, \nu, \upsilon_w, U_e), \quad \langle u'\upsilon'\rangle = F_2(x, y, \nu, \upsilon_w, U_e) \tag{1.1}$$

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Here υ is the kinematic viscosity, and the origin of the Cartesian system of coordinates is on the leading edge of the plate.

By virtue of the elliptical nature of the problem, the quantities occurring on the left-hand sides of equalities (1.1) can also depend on additional conditions, which are formulated below with respect to flow, for example, where separation of the boundary layer occurs in the case of injection. Only an attached boundary layer will subsequently be considered and the effect of the additional parameters on this is negligibly small.

We now introduce the boundary-layer thickness

$$\Delta = F_3(x, \nu, \upsilon_w, U_e) \tag{1.2}$$

into the treatment as a certain quantity which characterizes the transverse scale of the flow.

Now, expressing x and U_e from the first equation of (1.1) and Eq. (1.2) and substituting into the second equation of (1.1), we obtain

$$\langle u'\upsilon'\rangle = F_4\left(y, \nu, \upsilon_w, \Delta, \frac{\partial u}{\partial y}\right)$$

Applying dimensional consideration to this relation, we shall have

$$\langle u'\upsilon'\rangle = -\left(y\frac{\partial u}{\partial y}\right)^2 S(R_l, \beta, \eta); \quad R_l = \frac{y^2}{\nu}\frac{\partial u}{\partial y}, \quad \beta = \frac{\upsilon_w}{yR_l}\left(\frac{\partial u}{\partial y}\right)^{-1}, \quad \eta = \frac{y}{\Delta} \tag{1.3}$$

The local Reynolds number R_l is equal to the ratio of the characteristic values of the turbulent and molecular viscosities. Like the functions $F_1, \ldots, F_4$, the function S is universal for the class of flows being considered, which depend on the three parameters ν, υ_w, U_e. We will assume that it is continuous when $0 \le R_l \le \infty$, $-\infty \le \beta \le \infty$, $0 \le \eta < \infty$, differentiable within this domain, and that it satisfies the condition $S(\infty, 0, 0) \ne 0$. As was shown in [5], the last inequality is the condition for the existence of a logarithmic velocity profile in a boundary layer on an impermeable plate.

Relation (1.3) is an expression of the shear stress in terms of the mean velocity gradient. Since the effect of the transpiration rate must decrease on moving away from the wall, the parameter β was chosen such that the local Reynolds number R_l is in the denominator.

Similarly, for the normal turbulent stresses, we obtain

$$\langle u'^2\rangle = \left(y\frac{\partial u}{\partial y}\right)^2 S_1(R_l, \beta, \eta), \quad \langle \upsilon'^2\rangle = \left(y\frac{\partial u}{\partial y}\right)^2 S_2(R_l, \beta, \eta), \quad \langle w'^2\rangle = \left(y\frac{\partial u}{\partial y}\right)^2 S_3(R_l, \beta, \eta) \tag{1.4}$$

Here, S_1, S_2 and S_3 are certain universal functions. However, a continuity condition can only be formulated for the function S_2, which describes the behaviour of the Reynolds-tensor component associated with transverse velocity fluctuations. According to the theoretical representations in [8, 9], which are supported by measurements, this component behaves, close to the wall, in a similar manner to the shear stress. The quantity

$$S_2(\infty, 0, 0) = (\kappa/\sigma_2)^2 \tag{1.5}$$

therefore exists and is not equal to zero. Here, $\kappa = 0.41$ is von Kármán's constant and, according to experimental data, the constant $\sigma_2 = 0.95$ [4].

The behaviour of the components $\langle u'^2\rangle$ and $\langle w'^2\rangle$, which are associated with velocity fluctuations in directions parallel to the wall, has a more complex form, according to which

$$S_1(\infty, 0, \eta) = -A_1\kappa^2\ln\eta + O(1), \quad S_3(\infty, 0, \eta) = -A_3\kappa^2\ln\eta + O(1), \quad \eta \to 0 \tag{1.6}$$

where A_1 and A_3 are universal constants. According to experimental and theoretical data [10, 11], $A_1 = 1.1$ and $A_3 = 0.36$.

The conditions for the functions S, S_1, S_2, S_3 are formulated using the following physical assumptions [8, 9]: the viscosity is only essential in a thin region close to the wall (a viscous sublayer) and the outer scale (the thickness of the boundary layer) has no effect on the behaviour of the components $\langle u'\upsilon'\rangle$ and $\langle \upsilon'^2\rangle$ close to the wall but, by virtue of the considerable anisotropy of the near-wall vortices, it still has an effect on the components $\langle u'^2\rangle$ and $\langle w'^2\rangle$.

In the case being considered, the stream function of the averaged flow $\psi(x, y)$ satisfies the boundary-layer equation with zero pressure gradient

$$\begin{aligned}&\psi_y\psi_{xy} - \psi_x\psi_{yy} = (\nu\psi_{yy} - \langle u'v'\rangle)_y\\&x>0,\quad y = 0: \psi_y = 0,\quad \psi_x = -v_w;\quad y\to\infty:\quad \psi_y\to U_e,\quad \langle u'v'\rangle\to 0\end{aligned} \tag{1.7}$$

In problem (1.7) we now change to new variables using the formulae [12]

$$\psi = U_e\Delta\Psi(\xi,\eta),\ \Lambda(\xi) = \frac{dR_\Delta}{dR_x},\ \xi = \ln R_\Delta,\ \eta = \frac{y}{\Delta},\ R_x = \frac{U_e x}{\nu},\ R_\Delta = \frac{U_e\Delta}{\nu} \tag{1.8}$$

The logarithm of the Reynolds number based by the boundary-layer thickness and the normalized distance to the wall are chosen as the independent variables ξ and η. In addition to the dimensionless stream function $\Psi(\xi, \eta)$, a second unknown function $\Lambda(\xi)$ is introduced, which has the physical meaning of the rate of longitudinal change in the transverse scale Δ. Taking relation (1.3) into account, we obtain for the functions $\Psi(\xi, \eta)$ and $\Lambda(\xi)$

$$\Lambda[\Psi_\eta\Psi_{\xi\eta} - \Psi_{\eta\eta}(\Psi + \Psi_\xi)] = [(\eta\Psi_{\eta\eta})^2 S(R_l,\beta,\eta) + e^{-\xi}\Psi_{\eta\eta}]_\eta \tag{1.9}$$

$$R_l = e^{\xi}\eta^2\Psi_{\eta\eta},\quad \beta = B(\eta R_l\Psi_{\eta\eta})^{-1},\quad B = \frac{v_w}{U_e}$$

$$\xi>-\infty,\quad \eta = 0:\quad \Psi_\eta = 0,\quad \Lambda(\Psi + \Psi_\xi) = -B \tag{1.10}$$

$$\eta\to\infty:\quad \Psi_\eta\to 1,\quad \eta\Psi_{\eta\eta}\sqrt{S(R_l,\beta,\eta)}\to 0 \tag{1.11}$$

Relations (1.10) specify the conditions on the plate, and relations (1.1) specify the conditions on the outer edge of the boundary layer. In accordance with the order of the differential equation, two conditions are set for the velocity and shear stress on the outer edge.

We shall seek an asymptotic representation of the solution of problem (1.9)–(1.11) when $\xi\to\infty$. We introduce the small parameter ε and the new independent variable $\zeta = \varepsilon\xi$, $1/\zeta = O(1)$. The small parameter in the present problem is therefore a quantity which is the reciprocal of the logarithm of the Reynolds number, based on the characteristic transverse scale of the flow.

We specify the transpiration rate on the wall in the form

$$B = \varepsilon^2 b,\quad b = O(1) \tag{1.12}$$

that is, we shall assume the order of smallness of the magnitude of the transverse velocity on the wall to be quadratic in the parameter ε.

The asymptotic expansions of the solution with respect to the small parameter ε will have a different form for the different characteristic regions of the flow. Two such regions arise in a direction perpendicular to the wall: an outer region, where the molecular viscosity can be neglected in the boundary layer equation and the characteristic scale is the boundary-layer thickness, and a near-wall region, the characteristic scale of which is determined from the condition that the turbulent and viscous stresses are equal in order of magnitude.

2. THE NEAR-WALL REGION OF THE BOUNDARY LAYER

We introduce the following variables (wall variables)

$$y_+ = \frac{yU_e\varphi}{\nu},\quad u_+ = \frac{\psi_y}{U_e\varphi},\quad v_+ = \frac{v_w}{U_e\varphi},\quad \varphi = \sqrt{\frac{c_f}{2}} \tag{2.1}$$

We introduce the following variables (wall variables) where c_f is the skin-friction coefficient. We integrate Eq. (1.9) across the layer and change to the independent variables ξ and y_+. For the function $u_+(\xi, y_+)$, we obtain the equation

$$\left(y_+\frac{\partial u_+}{\partial y_+}\right)^2 S\left(R_l, \beta, \frac{y_+}{\varphi}e^{-\xi}\right) + \frac{\partial u_+}{\partial y_+} - 1 - v_+u_+ =$$

$$= \frac{\Lambda e^{-\xi}}{\varphi}\left(\frac{d\varphi}{d\xi}\int_0^{y_+} u_+^2 dy_+ + \frac{\partial}{\partial\xi}\int_0^{y_+} u_+^2 dy_+ - u_+\int_0^{y_+}\frac{\partial u_+}{\partial\xi}dy_+\right) \quad (2.2)$$

$$R_l = y_+^2\frac{\partial u_+}{\partial y_+}, \quad \beta = \frac{v_+}{y_+R_l}\left(\frac{\partial u_+}{\partial y_+}\right)^{-1}$$

Taking the limit when $\varepsilon \to 0$, $1/\zeta = O(1)$, $y_+ = O(1)$ we obtain the first-order ordinary differential equation

$$\left(y_+\frac{\partial U_+}{\partial y_+}\right)^2 S(R_l, \beta, 0) + \frac{dU_+}{dy_+} = 1 + v_+U_+, \quad U_+(0) = 0$$

$$R_l = y_+^2\frac{dU_+}{dy_+}, \quad \beta = \frac{v_+}{y_+R_l}\left(\frac{\partial U_+}{\partial y_+}\right)^{-1} \quad (2.3)$$

for the first term in the expansion $U_+(y_+)$ of the function $u_+(\xi, y_+)$.

The asymptotic form of the solution of Eq. (2.3) in the outer part of the near-wall region, that is outside the viscous sublayer, has the form [5]

$$\frac{2}{v_+}(\sqrt{1 + v_+U_+} - 1) = \frac{1}{\kappa}[\ln y_+ + C(v_+)] + O(y_+^{-\alpha}), \quad y_+ \to \infty, \quad \alpha > 0 \quad (2.4)$$

The quantity $\sqrt{S(\infty, 0, 0)}$ plays the role of the von Kármán constant and $C(v_+)$ is a certain function. At the same time, $C(v_+) = C_0 + O(v_+)$ when $v_+ \to 0$, where C_0 is the second constant in the logarithmic law for the velocity profile in the case of an impermeable wall [5]. According to experimental data, $C_0 = 2.05$.

The qualitative behaviour of the solution of the problem depends on the relation between the shear stress on the wall and in the outer region of the boundary layer. Three characteristic flow behaviours can be distinguished: moderate transpiration when the shear stress is of the same order of magnitude in the outer and near-wall regions, strong near-critical injection when the shear stress on the wall is much less than in the outer region and strong suction when the friction on the wall is, on the other hand, much greater than in the outer region of the boundary layer. The first type of flow behaviour is considered below. In this case, all the empirical constants and functions occurring in the solution can be determined from experimental data for an impermeable plate. Strong injection [13] and strong suction [14, 15] have already been investigated.

3. THE OUTER REGION OF THE BOUNDARY LAYER

The function $\Lambda(\xi)$, which specifies the rate of streamwise change in the boundary-layer thickness, is a small quantity. Hence, in the first approximation, the left-hand side of Eq. (1.9) can be equated to zero, whence it follows, when account is taken of the conditions on the outer edge, that the streamwise velocity is a constant across the layer. The perturbation of this constant quantity in the next approximation is of the order of $\Lambda(\xi)$.

We shall therefore seek a solution of problem (1.9), (1.10) in the outer region of the boundary layer, where $1/\eta = O(1)$, in the form

$$\Lambda(\xi) = \varepsilon\lambda_1(\zeta) + \varepsilon^2\ln\varepsilon\lambda_2(\zeta) + O(\varepsilon^2) \quad (3.1)$$

$$\Psi(\xi, \eta) = \Psi_w(\xi) + \eta + \Lambda(\xi)f(\zeta, \eta) + O(\varepsilon^2) \quad (3.2)$$

We put $f(\zeta, 0) = 0$; then $\Psi_w(\xi)$ is the value of the stream function on the wall. By virtue of relations (1.12) and (3.1), it follows from the second condition on the wall (1.10) that

$$\Psi_w(\xi)+\frac{d\Psi_w(\xi)}{d\xi} = O(\varepsilon) \tag{3.3}$$

Substituting expansions (3.1) and (3.2) into Eq. (1.9), taking account of relation (3.3), and taking the limit when $\varepsilon \to 0$, $1/\zeta = O(1)$, $1/\eta = O(1)$ we obtain the boundary-value problem (the notation $T(\eta) = \sqrt{S(\infty, 0, \eta)}$ has further been introduced)

$$[(\eta f_{\eta\eta}T)^2]_\eta + \eta f_{\eta\eta} = 0; \quad f(\zeta, 0) = f_\eta(\zeta, \infty) = 0, \quad \lim_{\eta\to\infty} \eta f_{\eta\eta}T \to 0 \tag{3.4}$$

for determining the function $f(\zeta, \eta)$, which enables us to conclude that the function $f(\zeta, \eta)$ can be chosen independently of the variable ζ. Moreover, for a specified order of magnitude of the transverse velocity on the wall, the solution of boundary-value problem (3.4) is of a universal nature, since it is independent of the conditions on the surface over which the flow occurs. In other words, the function $f(\eta)$ is exactly the same function for the cases of a transpired or an impermeable plate.

Integration of Eq. (3.4) gives

$$\eta f''T = D-\int_0^\eta \frac{d\eta}{2T}$$

where D is a constant. When account is taken of the condition for the shear stress, we shall have from this that

$$\eta f''T = \int_\eta^\infty \frac{d\eta}{2T}$$

and the improper integral on the right-hand side converges. Hence, the solution of the boundary-value problem (3.4) has the form

$$f'(\eta) = -\int_\eta^\infty \frac{d\eta_1}{2\eta_1 T(\eta_1)} \int_{\eta_1}^\infty \frac{d\eta_2}{T(\eta_2)} \tag{3.5}$$

Since $T(0) = \kappa$, from (3.5) we obtain the asymptotic form of the solution on the wall

$$f'(\eta) = \frac{F_1}{\kappa}(\ln\eta + A_0 - \ln F_1) + O(\eta^\alpha), \quad \eta \to 0, \quad F_1 \equiv \sqrt{-f(\infty)} = \int_0^\infty \frac{d\eta}{2T}, \quad \alpha > 0 \tag{3.6}$$

where A_0 is a certain constant.

Note that transformation (1.8) allows of a certain arbitrariness in the choice of the boundary-layer thickness so that this quantity can be not identical to the transverse scale occurring in the formulation of closure condition (1.3). We introduce the new transverse scale $\Delta_\circ = \Delta/\sigma$, where $\sigma(\zeta_\circ, \varepsilon)$ is an arbitrary function provided that the condition for a monotonic increase in the boundary-layer thickness is satisfied. We label all quantities, which are defined in the case of a boundary-layer thickness $\Delta_\circ$, with the subscript $\circ$. The equalities

$$\eta_\circ = \sigma\eta, \quad f_\circ(\eta_\circ) = \sigma^2 f(\eta) \tag{3.7}$$

follow from (3.5). According to these equalities the constant A_0 in the asymptotic representation (3.6) is independent of the method of defining the boundary-layer thickness.

4. MATCHING OF THE SOLUTIONS FOR THE OUTER AND NEAR-WALL REGIONS OF THE BOUNDARY LAYER

The solutions of the problems for the outer and near-wall regions contain two unknown functions: $\Lambda(\xi)$ and $\varphi(\xi)$, which can be found as the result of asymptotic matching [16]. In accordance with representations (3.1), (3.2) and (1.3), the estimate

$$\langle u'v'\rangle/U_e^2 \sim \Lambda^2(\xi) = O(\varepsilon^2)$$

holds for the turbulent shear stress in the outer region of the boundary layer. Considering the case of moderate transpiration when the shear stress is of the same order of magnitude on the wall and in the outer region, we will seek the skin-friction coefficient in the form

$$\varphi(\xi) = \varepsilon t_1(\zeta) + \varepsilon^2 \ln\varepsilon t_2(\zeta) + \varepsilon^2 t_3(\zeta) + \varepsilon^3 \ln^2\varepsilon t_4(\zeta) + \varepsilon^3 \ln\varepsilon t_5(\zeta) + O(\varepsilon^3) \tag{4.1}$$

For matching, we will base ourselves on relation (2.4), which holds in the near-wall region outside the viscous sublayer and the asymptotic representation for the streamwise velocity

$$\Psi_\eta = 1 + \frac{F_1}{\kappa}[\varepsilon\lambda_1(\zeta) + \varepsilon^2 \ln\varepsilon\lambda_2(\zeta)][\ln\eta + A_0 - \ln F_1 + O(\eta^\alpha)] + O(\varepsilon^2) \tag{4.2}$$

which is obtained from the solution for the outer region of the boundary layer when $\eta \to 0$ on the basis of representations (3.1) and (3.2) and asymptotic form (3.6). These two relations hold in the intermediate zone joining the two characteristic domains.

According to equalities (2.1), the relation between the wall variables y_+, U_+ and the outer variables η, Ψ_η has the form

$$y_+ = \varphi(\xi)e^{\zeta/\varepsilon}\eta, \quad U_+ = \frac{\Psi_\eta}{\varphi(\xi)} + O(\varepsilon e^{-\zeta/\varepsilon}), \quad v_+ = \frac{\varepsilon^2 b}{\varphi(\xi)} \tag{4.3}$$

An estimate of the order of the residual term in the second formula of (4.3) follows from the estimation of the right-hand side of Eq. (2.2) when account is taken of expansions (3.1) and (4.1).

In (2.4), we now change from the wall variables to the outer variables using formulae (4.3) and, when $1/\zeta = O(1)$, $1/\eta = O(1)$, we expand both sides of the equality with respect to the small parameter ε up to terms of the order of $\varepsilon\ln\varepsilon$ inclusive. We express the streamwise velocity using representation (4.2). Instead of the skin-friction coefficient, we substitute series (4.1) and, at the same time, assume that

$$t_1(\zeta) \neq 0, \quad t_1^2(\zeta) + b \neq 0 \tag{4.4}$$

Since $v_+ = O(\varepsilon)$ in this case, it is sufficient to use the estimate $C(v_+) = C_0 + O(\varepsilon)$ of the function $C(v_+)$ occurring in asymptotic form (2.4). We equate similar terms on the right-hand and left-hand sides of the equality obtained in this manner. As a result, we will have seven algebraic equations for the seven required functions $\lambda_1(\zeta)$, $\lambda_2(\zeta)$, $t_1(\zeta)$, ... , $t_5(\zeta)$, the solution of which is

$$\lambda_1(\zeta) = \frac{\kappa P(\zeta)}{F_1\zeta}, \quad \lambda_2(\zeta) = -\frac{\kappa Q(\zeta)}{F_1\zeta^2}, \quad t_1(\zeta) = \frac{\kappa Q(\zeta)}{\zeta}, \quad t_2(\zeta) = -\frac{\kappa P(\zeta)}{\zeta^2}$$

$$t_3(\zeta) = -\frac{\kappa P(\zeta)R(\zeta)}{\zeta^2}, \quad t_4(\zeta) = \frac{\kappa}{\zeta^3}, \quad t_5(\zeta) = \frac{\kappa}{\zeta^3}\left[\frac{P^2(\zeta)}{Q(\zeta)} + 2R(\zeta)\right] \tag{4.5}$$

$$P(\zeta) = 1 + b\left(\frac{\zeta}{2\kappa}\right)^2, \quad Q(\zeta) = 1 - b\left(\frac{\zeta}{2\kappa}\right)^2, \quad R(\zeta) = \ln\frac{\kappa F_1 Q(\zeta)}{\zeta} - A_0 + C_0$$

According to solution (4.5), the second inequality of (4.4) is equivalent to the inequality $\lambda_1(\zeta) \neq 0$ and inequalities (4.4) express the conditions that the leading terms of the expansion of the skin friction do not vanish during injection and that the function $\Lambda(\xi)$ does not vanish during suction.

We determine the connection between the variable ζ and the streamwise coordinates x from relations (1.8) and (3.1)

$$R_x = \int_{-\infty}^{\xi} \frac{de^\xi}{\Lambda(\xi)} = \frac{e^{\zeta/\varepsilon}}{\varepsilon}\left(\frac{1}{\lambda_1(\zeta)} - \varepsilon\ln\varepsilon\frac{\lambda_2(\zeta)}{\lambda_1^2(\zeta)} + O(\varepsilon)\right) \tag{4.6}$$

When account is taken of the expressions for the coefficients of series (4.5), expansions (3.1), (3.2), (4.1) and (4.6) give the required asymptotic representation of the solution for large values of $\ln R_\Delta$ which, however, has the serious drawback that, in the case of real Reynolds numbers, which determine the magnitude of the small parameter ε, the terms of asymptotic expansions (3.1), (4.1) and (4.6) do not decrease sufficiently rapidly.

We will make use of the freedom noted above in our choice of the boundary-layer thickness in order to transform the series to a more convenient form. To do this, we write expansions (3.1) and (4.1) in the following equivalent form

$$\Lambda(\xi) = \frac{\varepsilon\kappa P(\zeta)}{F_1\zeta} - \frac{\varepsilon^2\kappa Q(\zeta)}{F_1\zeta^2}[\ln\varepsilon + R(\zeta)] + O(\varepsilon^2)$$
$$\varphi(\xi) = \frac{\varepsilon\kappa Q(\zeta)}{\zeta} - \frac{\varepsilon^2\kappa P(\zeta)}{\zeta^2}[\ln\varepsilon + R(\zeta)] + \frac{\varepsilon^3\kappa}{\zeta^3}\left[\ln\varepsilon + R(\zeta) + \frac{P^2(\zeta)}{2Q(\zeta)}\right]^2 + O(\varepsilon^3) \tag{4.7}$$

Expansions (4.7) are identical to expansions (3.1) and (4.1), in which the values of coefficients (4.5) are set up, to written quantities of the order of ε^2 and ε^3, respectively.

We now introduce the new transverse scale $\Delta_\circ = \Delta/\sigma(\zeta_\circ, \varepsilon)$. The independent variables and the required functions corresponding to the two different scales are related to one another by the equalities

$$\zeta = \zeta_\circ + \varepsilon\ln\sigma, \quad \varphi(\xi) = \varphi_\circ(\xi_\circ), \quad \Lambda(\xi) = \Lambda_\circ(\xi_\circ)\left(\sigma + \varepsilon\frac{d\sigma}{d\zeta_\circ}\right) \tag{4.8}$$

The last equality was obtained in accordance with the definition of the function $\Lambda(\xi)$ (1.8).

Taking account of the fact that the value of the quantity F_1, which is defined by equality (3.6), changes according to formula (3.7) on changing to the new scale, we write expansion (4.7) for the new variable and put $\ln\sigma = -\ln\varepsilon - R(\zeta_\circ)$. Only the leading terms then remain in the new expansions and, according to equalities (4.8), we shall have

$$\zeta = \zeta_\circ - \varepsilon[\ln\varepsilon + R(\zeta_\circ)] \tag{4.9}$$

$$\Lambda(\xi) = \frac{\varepsilon\kappa P(\zeta_\circ)}{F_1\zeta_\circ}\left[1 + O\left(\frac{\varepsilon}{\zeta_\circ}\right)\right], \quad \varphi(\xi) = \frac{\varepsilon\kappa Q(\zeta_\circ)}{\zeta_\circ}\left[1 + O\left(\frac{\varepsilon^2}{\zeta_\circ^2}\right)\right] \tag{4.10}$$

Hence, an almost identical change of variables (4.9) reduces asymptotic expansions (4.7) to the simple single-term form (4.10). This can also be shown by direct substitution.

When account is taken of relations (4.9) and (4.10), integral (4.6) is transformed in the following manner

$$R_x = \int_{-\infty}^{\xi} \frac{de^\xi}{\Lambda(\xi)} = \frac{\zeta_\circ^2}{\varepsilon^2\kappa^2 P(\zeta_\circ)Q(\zeta_\circ)}\exp\left[\frac{\zeta_\circ}{\varepsilon} + A_0 - C_0 + O\left(\frac{\varepsilon}{\zeta_\circ}\right)\right] \tag{4.11}$$

An asymptotic solution of the problem is obtained in the boundary-layer approximation when the stream function satisfies Eq. (1.7). However, all of the results still hold if the system of complete Reynolds-averaged Navier–Stokes equations is used as the initial system when constructing expansions (3.1), (3.2) and (4.1). The terms in the Navier–Stokes equations which are not taken into account in boundary-layer theory only affect quantities of the order of ε^2 in expansions (3.1) and (3.2). The solution which has been constructed is therefore an asymptotic solution of the Reynolds-averaged Navier–Stokes equations which, when $\varepsilon \to \infty$, has the same accuracy as the boundary-layer equations.

5. THE UNIVERSAL VELOCITY-DEFECT LAW

Solving the second equation of (4.10) for $\varepsilon/\zeta_\circ$, we obtain

$$2\kappa\varepsilon/\zeta_\circ = (\Lambda_* + \varphi)[1 + O(\varepsilon_*)], \quad \Lambda_* = \sqrt{c_f/2 + B}, \quad \varepsilon_* = \sqrt{c_f + B} \tag{5.1}$$

Here, account has been taken of the fact that, by virtue of the inequality $\varepsilon_* \leq \Lambda_* + \varphi \leq \sqrt{2}\varepsilon_*$, we can write $O(\varepsilon_*)$ instead of $O(\Lambda_* + \varphi)$.

Substitution of expression (5.1) into the first relation of (4.10) gives

$$\Lambda = \frac{\Lambda_*}{F_1}[1 + O(\varepsilon_*)] \tag{5.2}$$

A similar expression has been obtained [17] (see also [4]) on the basis of dimensional considerations for the case of an impermeable plate.

We obtain the final form of the formulae for the streamwise velocity distribution in the outer region of the boundary layer and, also, of the Reynolds numbers, based on the boundary-layer thickness and distance from the leading edge of the plate, from relations (3.2), (4.9) and (4.11) taking relations (5.1) and (5.2) into account.

$$\frac{U_e - u}{U_e \Lambda_*} = -\frac{f'(\eta)}{F_1} + O(\varepsilon_*) \tag{5.3}$$

$$R_\Delta = \frac{E}{F_1 \varphi}, \quad R_x = \frac{E}{\varphi \Lambda_*}, \quad E = \exp\left[\frac{2\kappa}{B}(\Lambda_* - \varphi) + A_0 - C_0 + O(\varepsilon_*)\right] \tag{5.4}$$

Equality (5.3), obtained earlier in [7], is a universal velocity-defect law which extends the well-known relation for an impermeable plate to the case of transpiration.

The universal function $f'(\eta)$ can be defined using Coles' empirical formula for the velocity profile on an impermeable plate [18]

$$\frac{f'(\eta)}{F_1} = \frac{1}{\kappa}[\ln\eta - 0.55(1 + \cos(\pi\eta))], \quad 0 < \eta \le 1 \tag{5.5}$$

Here, the quantity Δ is chosen as the distance from the wall at which the streamwise component of the mean velocity differs from U_e by 0.5%. In this case, $F_1 = 1.55/\kappa, A_0 = 0.23$.

Integrating both sides of the equality (5.3) with respect to η from 0 to ∞, we will have

$$\delta^*/\Lambda_* = F_1\Delta[1 + O(\varepsilon_*)] \tag{5.6}$$

where δ^* is the displacement thickness of the boundary layer. For represent the results, it is convenient to use a transverse scale, defined by the equality

$$\Delta^* = \delta^*/\Lambda_* \tag{5.7}$$

Expression (5.7) is an extension of the boundary-layer thickness introduced by Clauser [19] to the case of a permeable plate. According to Clauser's data, $A_0 = 0.246$ which is extremely close to the value cited above which was calculated using Coles' formula (5.5).

In order to check the theoretical results, experiments [20–25] were chosen which had been carried out under conditions which correspond to a sufficient degree to the formulation of the theoretical problem. In particular, the level of turbulent fluctuations on the outer edge of the boundary layer in the experiments in [20–25] can be assumed to be small.

The notation

$$D = \frac{U_e - u}{U_e \Lambda_*}, \quad D_1 = \frac{2(U_e - u)}{U_e(\Lambda_* + \sqrt{c_f/2 + Bu/U_e})}$$

$$P = \frac{\nu \partial u/\partial y - \langle u'v' \rangle}{U_e^2 \Lambda_*^2}, \quad P_1 = \frac{\langle u'^2 \rangle}{U_e^2 \Lambda_*^2}, \quad P_2 = \frac{\langle v'^2 \rangle}{U_e^2 \Lambda_*^2}$$

is used in all the figures.

The experimental velocity profiles in the boundary layer on a plate with injection for different values of the parameter B and Reynolds numbers [20] are constructed in similarity variables (5.3) on the left-hand side of Fig. 1. The solid curves correspond to Coles' formula (5.5). The experimental data confirm the correctness of the velocity-defect law: in the outer region all the points lie on a single curve, which

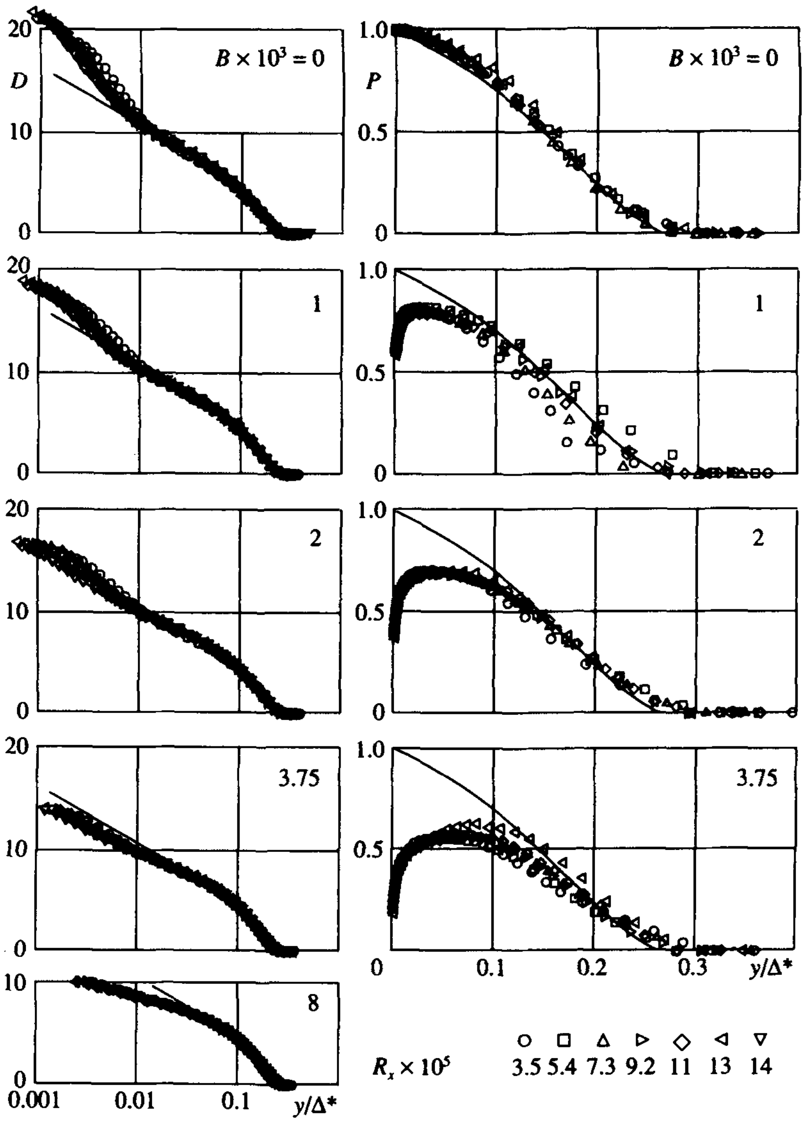

Fig. 1

corresponds to the velocity profile on an impermeable plate (the left-hand side of Fig. 1). At the same time, the interval, in which the proposed similarity relation is satisfied, depends very much on the injection parameter. For an impermeable plate ($B = 0$), the universal curve describes the velocity distribution through the whole thickness of the boundary layer apart from the viscous sublayer which is very close to the wall. As the rate of injection increases, the region of applicability of similarity law (5.3) contracts and, in the case of the value $B = 0.008$, which was the largest value of B in the series of experiments in [20], it is approximately 80% of the boundary-layer thickness.

Similarity law (5.3) must be satisfied at high Reynolds numbers. In the case of the data of Fig. 1, the minimum Reynolds number is 3.5×10^5. Velocity profiles, measured in [20] at lower Reynolds numbers ($R_x = 3.2 \times 10^4$, 1.6×10^5), are available: these data do not satisfy the velocity-defect law which has been established.

Velocity-defect law (5.3) enables one to formulate similarity rules for the distribution of the components of the Reynolds stress tensor. From relation (1.3) and Eq. (3.5) for the shear stress, we obtain

$$\frac{\nu \partial u/\partial y - \langle u'v' \rangle}{U_e^2 \Lambda_*^2} = 1 + \frac{\eta f'(\eta) - f(\eta)}{f(\infty)} + O(\varepsilon_*) \tag{5.8}$$

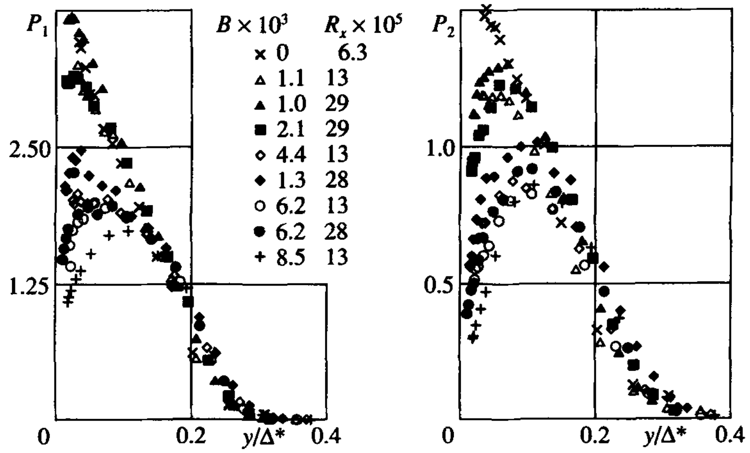

Fig. 2

From relations (1.4), for the normal turbulent stresses in the outer region we shall have

$$\frac{\langle u'^2 \rangle}{U_e^2 \Lambda_*^2} = \left[\frac{\eta f''(\eta)}{F_1}\right]^2 S_1(\infty, 0, \eta) + O(\varepsilon_*) \tag{5.9}$$

The relations for the components $\langle \upsilon'^2 \rangle$ and $\langle w'^2 \rangle$ are completely analogous to formula (5.9). There is a function on the right-hand side of equality (5.9) which specifies the distribution of the component of the Reynolds tensor in the boundary layer on an impermeable plate.

On the right-hand side of Fig. 1, the data from the measurements of the shear stress in [20], constructed in similarity variables (5.8), are compared with the calculated curve corresponding to the right-hand side of formula (5.8) in which the function $f'(\eta)$ is given by expression (5.5). In the case of an impermeable plate, the calculated curve and the experimental shear stress profiles are extremely close ($B = 0$, the right-hand side of Fig.1). This confirms the correctness of the approximate equation (3.5) obtained in this paper, since Coles' formula describes the experimental velocity profile quite well ($B = 0$, the left-hand side of Fig. 1). In the case of injection, use of variables (5.8) enables one to superpose the shear stress distributions obtained for a single value of the injection parameter but for different Reynolds numbers. (It is obvious that the anomalous behaviour of the three profiles corresponding to $B = 0.001$ and $3.5 \times 10^5 \leq R_x \leq 7.3 \times 10^5$, and of the single profile corresponding to $B = 0.00375$ and $R_x = 1.3 \times 10^6$ is due to the inaccuracy of the measurements). However, the experimental data only agree with the calculated curve in part of the boundary layer in the region which is adjacent to its outer edge; the length of this domain contracts appreciably as the injection rate increases.

In Fig. 2, the experimental profiles of the components of the Reynolds tensor associated with the streamwise velocity fluctuations (the left-hand side of Fig. 2) and the transverse velocity fluctuation (the right-hand side) [21] (also, see [22]) have been constructed for different values of B and R_x in similarity variables (5.9). It can be seen that the experimental data obtained over an extremely wide range of the injection parameter at different Reynolds numbers satisfy similarity rule (5.9) in region consisting of approximately three quarters of the boundary-layer thickness.

Velocity-defect law (5.3) can be written in the form

$$\frac{U_e - u}{U_e \Lambda_*} = -\frac{f'(\eta)}{F_1} + \varepsilon f_1(\zeta, \eta, \varepsilon), \quad f_1(\zeta, \eta, \varepsilon) = O(1), \quad \varepsilon \to 0$$

From this, for the integral boundary layer characteristics we obtain

$$\frac{\delta^*}{F_1 \Delta \Lambda_*} = 1 + \varepsilon I(\zeta, \varepsilon), \quad \frac{\theta}{F_1 \Delta \Lambda_*} = 1 - G\Lambda_* + \varepsilon I(\zeta, \varepsilon) + O(\varepsilon^2) \tag{5.10}$$

$$I(\zeta, \varepsilon) = \int_0^\infty \frac{f_1(\zeta, \eta, \varepsilon)}{F_1} d\eta, \quad G = \int_0^\infty \frac{f'^2 d\eta}{F_1^3} \tag{5.11}$$

Fig. 3

Here θ is the momentum thickness. According to equalities (3.7), the second integral of (5.11) is independent of the method of defining the boundary-layer thickness. The quantity G is known as the Clauser shape factor [19], and, according to the data from measurements in a boundary layer on an impermeable plate, its numerical value lies in the range from 6.4 to 7.1; calculations based on Coles' formula (5.5) give $G = 6.61$.

The similarity rule for the boundary-layer shape factor follows from relations (5.10)

$$(1-\theta/\delta^*)/\Lambda_* = G + O(\varepsilon_*) \tag{5.12}$$

which extends the well-known representation for an impermeable plate [19] to the case of transpiration.

The values of the shape factor for a boundary layer on a plate with injection, which have been transformed in accordance with formula (5.12), are constructed in Fig. 3 using the experimental data from [20, 23, 24]. As a whole, the data from the measurements correspond to similarity rule (5.12), although the value of the constant, around which the experimental points are mainly concentrated, is 20% less than the value which is characteristic of a boundary layer without injection.

6. COMPOSITE EXPANSIONS FOR VELOCITY AND REYNOLDS STRESS PROFILE

Velocity-defect law (5.3) is correct in the outer region but it is not satisfied in the near-wall region where the velocity distribution is described by the function $U_+(y_+)$. Outside the viscous sublayer, this function satisfies similarity law (2.4). A composite expansion, constructed using relations (2.4) and (5.3), enables one to obtain a representation of the velocity profile in the whole of the boundary layer apart from the viscous sublayer.

We will rewrite similarity law (2.4) in the outer variables taking account of the expression for the boundary-layer thickness (5.4).

$$\sqrt{1-qD} = 1+\frac{q}{2}\Phi + O(q^2), \quad \Phi(\eta) = \frac{1}{\kappa}(\ln\eta + A_0 - \ln F_1), \quad q = \frac{B}{\Lambda_*} \tag{6.1}$$

On the basis of relations (1.12) and (4.1), $q = O(\varepsilon)$ in the case being considered of moderate transpiration. Solving Eq. (6.1) assuming that $q^\Phi = O(1)$, we obtain

$$D = -\Phi - \frac{q}{4}\Phi^2 + O(q) \tag{6.2}$$

Equalities (6.2) and (5.3) are now considered everywhere outside the viscous sublayer. We form the composite expansion for the velocity defect D in the form of a sum of expansions for the two regions, after subtracting their common part [16]

$$D = -\frac{f'}{F_1} - \frac{q}{4}\Phi^2 + O(\varepsilon_*) \tag{6.3}$$

We now replace $\Phi(\eta)$ by the function $f'(\eta)/F_1$ which, according to (3.6), has the same asymptotic form on the wall. This does not change the order of accuracy of representation (6.3), but allows the physical condition that the velocity defect should vanish on the outer edge of the boundary layer to be satisfied. Hence, finally we will have

$$\frac{U_e-u}{U_e\Lambda_*} = -\frac{f'(\eta)}{F_1} - \frac{qf'^2(\eta)}{4F_1^2} + O(\varepsilon_*) \tag{6.4}$$

Here, the estimate $O(\varepsilon_*)$ uniformly holds everywhere outside the viscous sublayer.

Substituting expressions (6.4) into relations (1.3) and (1.4) we obtain the composite expansions for the Reynolds stresses, which represent these quantities solely in terms of functions which are known in the case of flow over an impermeable plate,

$$\frac{\sqrt{\nu\partial u/\partial y - \langle u'v'\rangle}}{U_e\Lambda_*} = \left[1 + \frac{\eta f'(\eta) - f(\eta)}{f(\infty)}\right]^{1/2}\left[1 + \frac{qf'(\eta)}{2F_1}\right] + O(\varepsilon_*) \tag{6.5}$$

$$\frac{\sqrt{\langle v'^2\rangle}}{U_e\Lambda_*} = \frac{\eta f''(\eta)}{F_1}\left[1 + \frac{qf'(\eta)}{2F_1}\right]\sqrt{S_1(\infty, 0, \eta)} + O(\varepsilon_*) \tag{6.6}$$

The formulae for the components $\langle u'^2\rangle$ and $\langle w'^2\rangle$ are completely analogous to the notation of (6.6). Expressions (6.5) and (6.6) describe the profiles of the shear stress and of the root-mean-square transverse velocity fluctuation everywhere outside the viscous sublayer. The region of applicability of the similarity laws for streamwise and spanwise fluctuations is narrower and, close to the wall, is restricted to the region where asymptotic representations (1.6) hold.

Using formulae (1.5) and (1.6), and asymptotic representation (3.6), we obtain the following expansions close to the wall for the components of the Reynolds stresses

$$\sqrt{\nu\frac{\partial u}{\partial y} - \langle u'v'\rangle} = \frac{v_w}{2\kappa}\ln\eta + O(1), \quad (\langle u'^2\rangle)^{1/3} = \frac{v_w}{|v_w|}\left(\frac{A_1 v_w^2}{4\kappa^2}\right)^{1/3}\ln\eta + O(1)$$
$$\sqrt{\langle v'^2\rangle} = \frac{v_w}{2\kappa\sigma_2}\ln\eta + O(1), \quad (\langle w'^2\rangle)^{1/3} = \frac{v_w}{|v_w|}\left(\frac{A_3 v_w^2}{4\kappa^2}\right)^{1/3}\ln\eta + O(1), \quad \eta\to 0 \tag{6.7}$$

All the constants in expansions (6.7) are known from experimental data in the case of an impermeable plate. It can be seen that there is a region close to the wall in which the root of the shear stress and the root-mean-square transverse velocity fluctuation behave as the logarithm of the distance to the wall and the root-mean-square streamwise and spanwise fluctuations behave as the logarithm to the power of 3.2.

Another form of the composite expansion for the velocity profile is obtained on solving Eq. (6.4) for $f'(\eta)$

$$\frac{2(U_e-u)}{U_e(\Lambda_* + \sqrt{c_f/2 + Bu/U_e})} = -\frac{f'(\eta)}{F_1} + O(\varepsilon_*) \tag{6.8}$$

Integration of equality (6.4) from the wall to the outer edge of the boundary layer gives a relation which is analogous to (5.6)

$$\Delta^{**} \equiv \frac{\delta^*}{\Lambda_* - GB/4} = F_1\Delta[1 + O(\varepsilon_*)]$$

where G is the Clauser shape factor (5.11). The transverse scale Δ^{**}, which is improved compared to (5.7), therefore corresponds to similarity law (6.8).

Relation (6.8) was proposed for the first time in [26] as a semiempirical dependence.

According to their construction, asymptotic representations (6.8) and (6.5) for the velocity and shear stress profiles hold everywhere outside the viscous sublayer. This is confirmed by Fig. 4, in which the

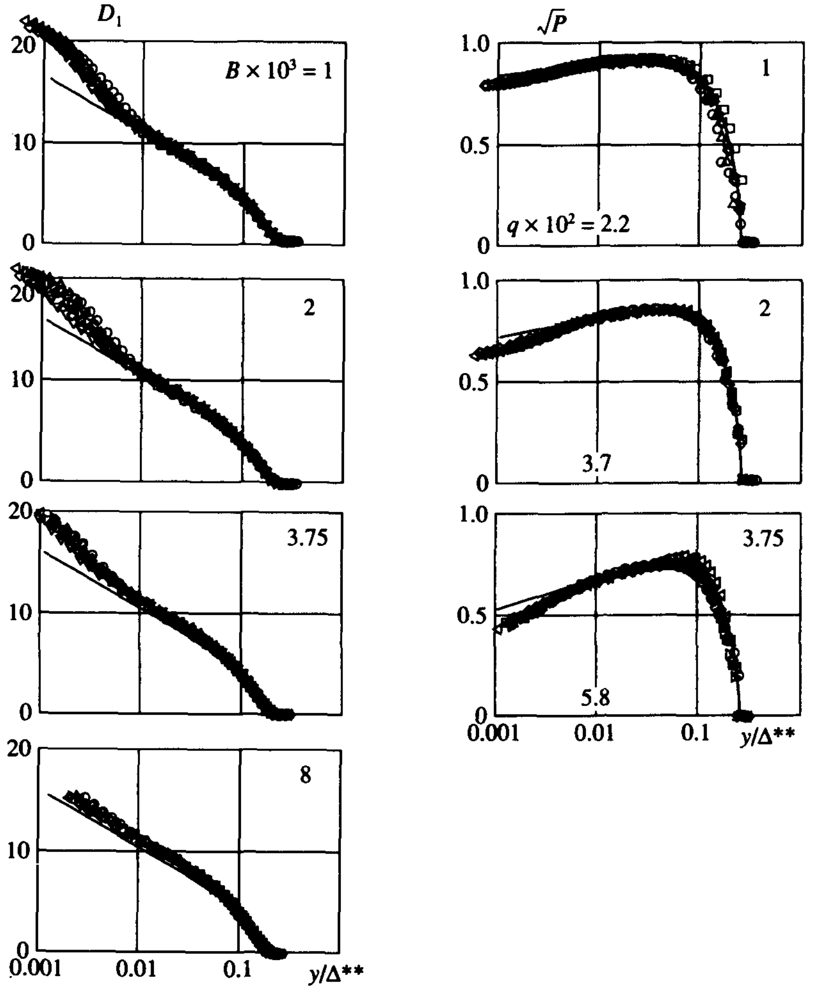

Fig. 4

same experimental data for the velocity distribution (the left-hand side of Fig. 4) and shear stress (the right-hand side) are constructed in variables (6.8) and (6.5). The velocity profiles [25], obtained not only in the case of injection but also suction, are presented in Fig. 5 for different values of the parameter B and Reynolds numbers.

It is now clear from the left-hand side of Fig. 4 and Fig. 5 that the experimental values of the velocity follow the universal curve almost along the whole boundary-layer thickness irrespective of the dependence on the magnitude of the transpiration parameter. Only points belonging to the viscous sublayer deviate from the universal distribution.

Similarity, it can be seen from the right-hand side of Fig. 4 that the two-term formula (6.5) provides a good description of the shear stress profile outside the viscous sublayer for all values of the injection parameter; the calculated curves are constructed for the values of q indicated in Fig. 4. For constant B, the parameter q changes slightly along the length of the plat and, therefore, in each case, its value has been taken as the mean for the given range of Reynolds numbers. It can be seen that, in accordance with formula (6.5), the experimental shear stress profiles have a distinct logarithmic segment close to the wall.

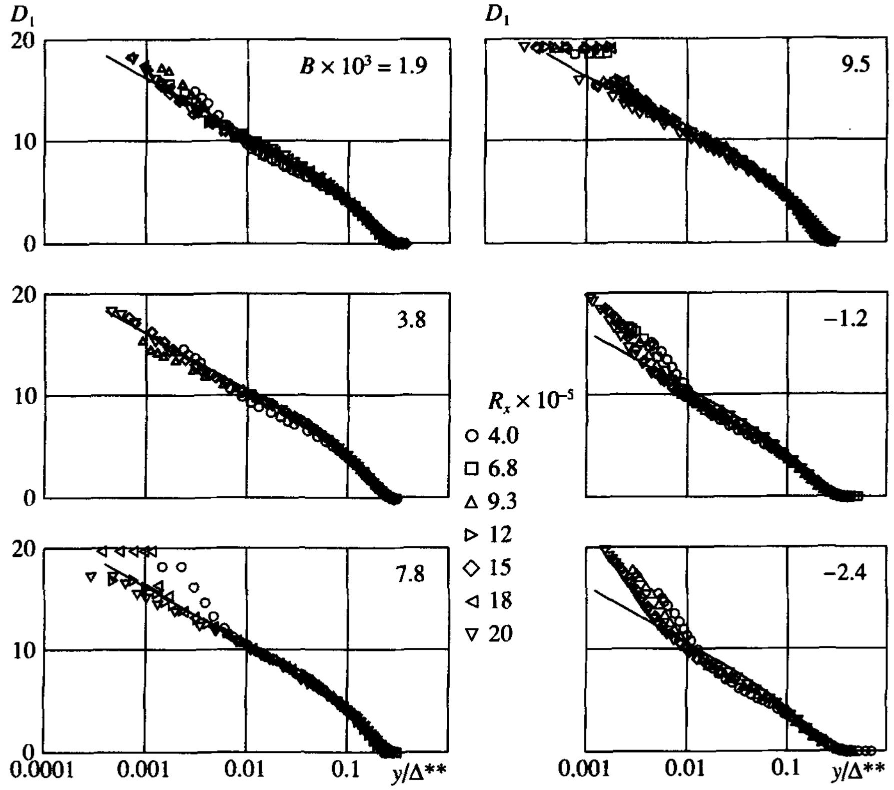

Fig. 5

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REFERENCES

1. IZAKSON, A. A., A formula for the velocity distribution close to a wall. *Zh. Eksper Teor. Fiz.*, 1937, 7, 7, 919–924.
2. MILLIKAN, C. G., A critical discussion of turbulent flows in channels and circular tubes. *Proc. 5th Intern. Congr. Appl. Mech.* Wiley. New York 1939, 386–392.
3. MISES, R. Von., Some remarks on the laws of turbulent motion in tubes. Th. von Kármán Anniversary volume. Calif. Inst. Techn. Press, Pasadena. 1941, 317–327.
4. KADER, B. A. and YAGLOM, A. M., Similarity laws for near-wall turbulent flows. In *Adverse in Science and Technology, Ser. Fluid Mechanics.* VINITI, Moscow, 1980, **15**, 81–155.
5. VIGDOROVICH, I. I., Similarity laws for the velocity and temperature distribution in the near-wall region of a turbulent boundary layer with injection and suction. *Dokl. Ross. Akad. Nauk*, 2003, **392**, 3, 340–345.
6. VIGDOROVICH, I. I., The relation between the hydrodynamic parameters of two different turbulent near-wall flows. *Dokl. Ross, Akad.Nauk*, 2004, **396**, 1, 47–51.
7. VIGDOROVICH, I. I., Universal velocity-defect law for a turbulent boundary layer on a plate with injection and suction. *Dokl. Ross. Akad. Nauk*, 1993, **331**, 4, 443–448.
8. TOWNSEND, A. A., *The Structure of Turbulent Shear Flow*, 2nd ed. Cambridge University Press, 1976.
9. PERRY, A. E., HENBEST, S. and CHONG, M. S., A theoretical and experimental study of wall turbulence. *J. Fluid Mech.* 1986, 165, 163–199.
10. FERNHOLZ, H. H. and FINLEY, P. J., The incompressible zero-pressure-gradient turbulent boundary layer: An assessment of the data. *Prog. Aerospace Sci.*, 1996, **32** 245–311.
11. SPALART, P. R., Direct simulation of a turbulent boundary layer up to $Re_\theta = 1410$. *J. Fluid Mech.*, 1988, **187**, 61–98.
12. VIGDOROVICH, I. I., Asymptotic investigation of a turbulent boundary layer on a flat plate at high Reynolds numbers. *Izv. Ross. Akad. Nauk. MZhG*, 1993, 4, 106–177.
13. VIGDOROVICH, I. I., Universal law of friction for a turbulent boundary layer on a plate with injection. *Dokl. Ross. Akad. Nauk*, 1994, **337**, 1, 39–43.
14. VIGDOROVICH, I. I., Universal law of friction for a boundary layer on a plate with suction, *Dokl. Ross. Akad. Nauk*, 1997, **356**, 1, 42–46.

15. VIGDOROVICH, I. I., Turbulent boundary layer on a plate with intense suction. *Izv. Ross. Akad. Nauk. MZhG*, 1999, 3, 61–76.
16. Van DYKE, M., *Perturbation Methods in Fluid Mechanics*. Academic Press. New York, 1964.
17. LANDAU, L. D. and LIFSHITZ, E. M., *Fluid Dynamics*. Pergamon Press, Oxford, 1987.
18. COLES, D., The law of the wake in the turbulent boundary layer. *J. Fluid Mech.*, 1856, **1**; 2, 191–226.
19. CLAUSER, F. H., The turbulent boundary layer. *Adv. Appl. Mech.*, Vol. 4. Academic Press, New York, 1956.
20. ANDERSEN, P. S., KAYS, W. M. and MOFFAT, R. J., The turbulent boundary layer on a porous plate: An experimental study of the fluid mechanics for adverse free-stream pressure gradient. Rep. HMT-15. Stanford Univ., 1972.
21. ALP, E., Measurements on a transpired turbulent boundary layer and exist of a mixing length model employing near-wall characteristic time scale. PhD thesis. University of Waterloo, Canada, 1978.
22. ALP, E. and STRONG, A. B., Measurements of characteristic time scales of the turbulent boundary layer with mass transfer. *Intern. J. Heat and Mass Transfer*, 1981, **24**, 3, 521–531.
23. SIMPSON, R. L., MOFFAT, R. J. and KAYS, W. M., The turbulent boundary layer on a porous plate: experimental skin friction with variable injection and suction. *Intern. J. Heat and Mass Transfer*, 1969, **12**, 7, 771–789.
24. DEPOOTER, K., BRUNDRETT, E. and STRONG, A. B., The calibration of Preston tubes in transpired turbulent boundary layers. *Trans. ASME. J. Fluids Eng.*, 1978, **100**, 1, 10–16.
25. SIMPSON, R. L., MOFFATT, R. M. and KAYS,W. M., The turbulent boundary layer on a porous plate: an experimental study of the fluid dynamics with injection and suction. Rep. HMT-2. Stanford Univ., 1967.
26. STEVENSON, T. N., Turbulent boundary layers with transpiration. *AIAA J.* 1964, **2**, 8, 1500–1502.

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